J. DIFFERENTIAL GEOMETRY
4 (1970) 169-185

TEICHMÜLLER THEORY FOR SURFACES WITH BOUNDARY

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1. Introduction

(A) Recently Earle and Eells [9] determined the homotopy types of the diffeomorphism groups of closed surfaces. Here similar methods are applied to compact surfaces with boundary. As in [9] we form a principal fibre bundle whose total space consists of the smooth conformal structures on the surface, whose base is the reduced Teichmüller space, and whose structure group is a group of diffeomorphisms of the surface. Again, as in [9], we rely on a new theorem about continuous dependence on parameters for solutions of Beltrami equations. The proof of that theorem is given in § 8. The remainder of the paper can be read independently of § 8, but the reader will find it helpful to consult [9]. Fuller accounts of Teichmüller theory may be found in [2], [5], [10], [13].

(B) Now we shall state our main theorems. Let X be a smooth (C^∞) surface with boundary, and denote by $\mathscr{D}(X)$ the topological group of all diffeomorphisms of X, with the C^∞-topology of uniform convergence on compact sets of all differentials. $\mathscr{D}_0(X)$ is the subgroup consisting of the diffeomorphisms which are homotopic to the identity and map each boundary curve onto itself, preserving orientation. We shall find later that $\mathscr{D}_0(X)$ is the arc component of the identity in $\mathscr{D}(X)$.

We denote by $\mathscr{M}(X)$ the space of smooth conformal structures on X, again with the C^∞ topology. There is a natural action

$$\mathscr{M}(X) \times \mathscr{D}(X) \to \mathscr{M}(X)$$

defined by letting $\mu \cdot f$ be the pullback of the metric μ by the diffeomorphism f.

Theorem. *Assume that X is compact and orientable and that the Euler characteristic $e(X)$ is negative. Then*

(a) *$\mathscr{M}(X)$ is a contractible Fréchet manifold,*

(b) *$\mathscr{D}_0(X)$ acts freely, continuously, and properly on $\mathscr{M}(X)$,*

(c) *the quotient map*

Communicated by James Eells, Jr., May 2, 1969, and in revised form, July 30, 1969. This research was partially supported by the National Science Foundation grants GP-8219 and GP-8413.

$$\Phi: \mathcal{M}(X) \to \mathcal{M}(X)/\mathcal{D}_0(X) = \mathcal{T}^*(X) \tag{1.1}$$

(with the quotient topology on $\mathcal{T}^*(X)$) defines a principal $\mathcal{D}_0(X)$-fibre bundle.

$\mathcal{T}^*(X)$ is the *reduced Teichmüller space* of the bordered surface X. The theorem will be proved in §§ 3 and 4.

(C) Because of Teichmüller's theorem [6], $\mathcal{T}^*(X)$ in (1.1) is a cell, and the fibre bundle (1.1) is trivial. Since $\mathcal{M}(X)$ is contractible, the structure group $\mathcal{D}_0(X)$ is contractible as well.

Theorem. *Let X be any smooth compact surface with boundary.*

(a) *If X is the closed disk, annulus, or Möbius strip, then $\mathcal{D}_0(X)$ has $SO(2)$ as strong deformation retract.*

(b) *In all other cases, $\mathcal{D}_0(X)$ is contractible.*

The cases not covered by Theorem 1B and Teichüller's theorem (X not orientable or $e(X) \geq 0$) are discussed in §§ 2, 5 and 6. In all cases Teichmüller theory and the theory of Beltrami equations play central roles in our proofs.

Let $C(X)$ be the homeomorphism group of X, with compact-open topology. Hamstrom [11] has computed the homotopy groups of the identity component of $C(X)$; they coincide with the homotopy groups of $\mathcal{D}_0(X)$ as computed from the above theorem.

(D) Let $\mathcal{D}_1(X)$ be the closed subgroup of $\mathcal{D}_0(X)$ consisting of the $g \in \mathcal{D}_0(X)$, which are homotopic to the identity modulo ∂X (fixing ∂X pointwise). In § 7 we prove the following.

Theorem. *Let X be a smooth compact surface with boundary. Then the group $\mathcal{D}_1(X)$ is contractible.*

As one would expect, Theorem 1D is a rather easy consequence of Theorem 1C. Moreover, our argument in § 7 is reversible and could be used to obtain Theorem 1C from Theorem 1D if a direct proof of the latter were available.

2. Beltrami equations

(A) Let D be a subregion of $\boldsymbol{R}^2$, bounded by smooth curves. If I is an open subset of ∂D, then $D \cup I$ is a smooth surface with boundary. The Fréchet space $C^\infty(D \cup I, \boldsymbol{C})$ is the vector space of smooth complex valued functions on $D \cup I$ with C^∞ topology. The subset $C^\infty(D \cup I, \Delta)$ consists of the smooth maps $D \cup I$ into the unit disk $\Delta = \{z \in \boldsymbol{C};\ |z| < 1\}$. As usual, we identify that subset with the space $\mathcal{M}(D \cup I)$ of smooth conformal structures on $D \cup I$ by assigning to each $\mu: D \cup I \to \Delta$ the conformal structure represented by

$$ds = |dz + \mu(z)d\bar{z}|\,, \qquad z \in D \cup I\,. \tag{2.1}$$

The zero function corresponds to the usual conformal structure on $D \cup I$.

Give $D \cup I$ the structure (2.1) and $\boldsymbol{C}$ its usual conformal structure. The orientation preserving diffeomorphism $w: D \cup I \to w(D \cup I) \subset \boldsymbol{C}$ is a conformal equivalence if and only if it satisfies Beltrami's equation

(2.2)
$$w_{\bar{z}} = \mu w_z \ ,$$

where

$$w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial x}\right) , \qquad w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) .$$

(B) Now let D be the upper half plane $\mathscr{U} = \{z \in \boldsymbol{C};\ \mathrm{Im}\, z > 0\}$, and suppose that $|\mu(z)| \leq k < 1$ in $\mathscr{U}$. There is a unique solution w_μ of (2.2), which is a homeomorphism of the closure of $\mathscr{U}$ onto itself and leaves $0, 1, \infty$ fixed [7, p. 277]. If $\mu \in C^\infty(\mathscr{U} \cup I)$, then w_μ is a diffeomorphism of $\mathscr{U} \cup I$ onto its image. Further,

Theorem. *For each $k < 1$, the map $\mu \mapsto w_\mu$ is a homeomorphism of the set of $\mu \in \mathscr{M}(\mathscr{U} \cup I)$ with $\sup\{|\mu(z)|;\ z \in \mathscr{U} \cup I\} \leq k < 1$ onto its image in $C^\infty(\mathscr{U} \cup I, \boldsymbol{C})$.*

That theorem, which we prove in § 8, is fundamental in all that follows. The easier case when I is empty was used in [9].

(C) As a corollary of Theorem 2B, we shall prove the simplest case of Theorem 1C. Let X be the closed unit disk, and X_0 its interior. Let $\mathscr{D}_0(X;\ 1, i, -1)$ be the topological group of all diffeomorphisms of X, which fix the points $1, i$, and -1. Define conformal maps h_1 and h_2 from $\mathscr{U}$ onto X_0 by

$$h_1(z) = \frac{i - z}{i + z} , \quad h_2^{-1}(h_1(z)) = f(z) = \frac{1}{1 - z} .$$

Each μ in $\mathscr{M}(X)$ induces conformal structures $\mu_1, \mu_2 \in \mathscr{M}(\mathscr{U} \cup \boldsymbol{R})$ via the maps h_1 and h_2. Explicitly

$$\mu(h_1(z))\overline{h_1'(z)}/h_1'(z) = \mu_1(z) \ ,$$

and

(2.3)
$$\mu_1(z) = \mu_2(f(z))\overline{f'(z)}/f'(z) \ , \quad z \in \mathscr{U} \cup \boldsymbol{R} \ .$$

Let $w_i = w_{\mu_i}, i = 1, 2$. Then $f^{-1} \circ w_2 \circ f = w_1$ because of (2.3); that is,

$$f_\mu = h_1 \circ w_1 \circ h_1^{-1} = h_2 \circ w_2 \circ h_2^{-1} \in \mathscr{D}_0(X;\ 1, i, -1) \ .$$

Of course f_μ is the unique element of $\mathscr{D}_0(X;\ 1, i, -1)$ to satisfy the Beltrami equation $f_{\bar{z}} = \mu f_z$.

Theorem. *The map $\mu \mapsto f_\mu$ is a homeomorphism from $\mathscr{M}(X)$ onto $\mathscr{D}_0(X;\ 1, i, -1)$.*

Proof. Apply Theorem 2B to w_1 and w_2, noting that if $\mu_n \to \mu$ in $\mathscr{M}(X)$, there is a number $k < 1$ such that

$$\sup\{|\mu_n(z)|;\ z \in X\} \leq k \qquad \text{for all } n \ .$$

Corollary. *The rotation group $SO(2)$ is a strong deformation retract of $\mathscr{D}_0(X)$.*

Proof. $\mathscr{D}_0(X)$ is homeomorphic to $\mathscr{D}_0(X; 1, i, -1) \times \operatorname{Aut} X$, where $\operatorname{Aut} X$ is the holomorphic automorphism group of X. But $\mathscr{D}_0(X; 1, i, -1)$ is homeomorphic to the contractible space $\mathscr{M}(X)$, and $\operatorname{Aut} X$ has $SO(2)$ as a strong deformation retract.

3. The proper action of $\mathscr{D}_0(X)$, $e(X) < 0$

(A) Let G be the conformal automorphism group of the upper half plane $\mathscr{U}$. Endowed with the compact-open topology, G is a Lie group; its identity component G_0 consists of the Möbius transformations

$$A(z) = (az + b)(cz + d)^{-1} ; \qquad a, b, c, d \in \boldsymbol{R};\ ad - bc = 1 .$$

G is generated by G_0 and the transformation $J(z) = -\bar{z}$.

(B) Let X be a compact smooth oriented surface with boundary, and X_0 its interior. Each $\mu \in \mathscr{M}(X)$ determines a complex structure on X_0. If the Euler characteristic $e(X)$ is negative, there is a holomorphic covering map $\pi: \mathscr{U} \to X_0$. The cover group Γ is a discrete subgroup of G_0; such groups are called Fuchsian. Since X has boundary, Γ is a group of the second kind. That means the limit set $L(\Gamma)$ is a Cantor set in $\boldsymbol{R} \cup \{\infty\}$. The complement of $L(\Gamma)$ is an open set I in $\boldsymbol{R}$. Γ acts freely and properly discontinuously on I; π extends to a covering $\pi: \mathscr{U} \cup I \to X$.

From π we obtain the induced map $\pi^*: \mathscr{M}(X) \to \mathscr{M}(\mathscr{U} \cup I)$, whose image $\mathscr{M}(\Gamma)$ consists of the Γ-invariant conformal structures on $\mathscr{U} \cup I$. These are the $\mu \in C^\infty(\mathscr{U} \cup I, \Delta)$ which satisfy

$$(3.1) \qquad (\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu \qquad \text{for all } \gamma \in \Gamma .$$

Let $A^1(\Gamma)$ be the Fréchet space of all $\mu \in C^\infty(\mathscr{U} \cup I, C)$, which satisfy (3.1).

Proposition. *$\mathscr{M}(\Gamma)$ is the convex open set of $\mu \in A^1(\Gamma)$ with $\sup\{|\mu(z)|;\ z \in \mathscr{U} \cup I\} < 1$, and the map $\pi^*: \mathscr{M}(X) \to \mathscr{M}(\Gamma)$ is a homeomorphism.*

Corollary. *$\mathscr{M}(X)$ is a contractible Fréchet manifold.*

The proofs are the same as the corresponding ones in § 5A of [9]. Note that the corollary is part (a) of Theorem 1B. Part (b) will be proved in the remainder of § 3.

(C) Let $\mathscr{D}(\mathscr{U} \cup I)$ be the metrizable topological group of all diffeomorphisms of $\mathscr{U} \cup I$, with C^∞ topology, and $\mathscr{D}(\Gamma)$ the normalizer of Γ in $\mathscr{D}(\mathscr{U} \cup I)$. Then $\pi_*(f) \circ \pi = \pi \circ f$ defines a continuous epimorphism $\pi_*: \mathscr{D}(\Gamma) \to \mathscr{D}(\mathscr{U} \cup I)$, and the kernel of π_* is Γ.

Lemma. *π_* is an open map.*

The proof is given in § 5B of [9], except that we use here the hyperbolic metric on $\mathscr{U} \cup I \cup \mathscr{U}^* = C - L(\Gamma)$.

Corollary. π_* *induces an isomorphism between the topological groups* $\mathscr{D}(\Gamma)/\Gamma$ *and* $\mathscr{D}(X)$.

Now let $\mathscr{D}_0(\Gamma)$ be the centralizer of Γ in $\mathscr{D}(\Gamma)$. Recall that $\mathscr{D}_0(X)$ is the set of g in $\mathscr{D}(X)$, which are homotopic to the identity.

Proposition. $\pi_*: \mathscr{D}_0(\Gamma) \to \mathscr{D}_0(X)$ *is an isomorphism of topological groups.*

Proof. It is proved in [6, pp. 98–100] that $\pi_*(\mathscr{D}_0(\Gamma)) = \mathscr{D}_0(X)$. The kernel of $\pi_*: \mathscr{D}(\Gamma) \to \mathscr{D}(X)$ is Γ. Since Γ is a free group on at least two generators, $\mathscr{D}_0(\Gamma) \cap \Gamma$ is trivial and $\pi_*: \mathscr{D}_0(\Gamma) \to \mathscr{D}_0(X)$ is bijective. For the proof that π_*^{-1} is continuous, see § 5B of [9].

(D) Using π, we transfer the action of $\mathscr{D}(X)$ on $\mathscr{M}(X)$ to an action of $\mathscr{D}(\Gamma)$ on $\mathscr{M}(\Gamma)$, given by

$$(3.2) \qquad (\pi^*\mu)\cdot g = \pi^*(\mu\cdot\pi_* g)\ , \qquad g \in \mathscr{D}(\Gamma), \mu \in \mathscr{M}(X)\ .$$

Proposition.

1. *The action* $\mathscr{M}(\Gamma) \times \mathscr{D}(\Gamma) \to \mathscr{M}(\Gamma)$ *defined by* (3.2) *is continuous.*
2. *The isotropy group of* $0 \in \mathscr{M}(\Gamma)$ *is* $\mathscr{D}(\Gamma) \cap G$, *the normalizer of* Γ *in* G.
3. $\Gamma = \{g \in \mathscr{D}(\Gamma);\ g$ *acts trivially on* $\mathscr{M}(\Gamma)\}$.
4. $\mathscr{D}_0(\Gamma)$ *acts freely on* $\mathscr{M}(\Gamma)$.

Corollary. *The action of* $\mathscr{D}(X)$ *on* $\mathscr{M}(X)$ *is continuous and effective, and* $\mathscr{D}_0(X)$ *acts freely.*

The proofs are given in § 5C of [9].

(E) **Proposition.** $\mathscr{D}_0(X)$ *acts properly on* $\mathscr{M}(X)$.

Proof. We prove the equivalent proposition that $\mathscr{D}_0(\Gamma)$ acts properly on $\mathscr{M}(\Gamma)$. Since the action is free, we need to prove merely that the map $\Theta: \mathscr{M}(\Gamma) \times \mathscr{D}_0(\Gamma) \to \mathscr{M}(\Gamma) \times \mathscr{M}(\Gamma)$ given by $\Theta(\mu, f) = (\mu, \mu\cdot f)$ is closed. Let $K \subset \mathscr{M}(\Gamma) \times \mathscr{D}_0(\Gamma)$ be a closed set, and $((\mu_n, \mu_n\cdot f_n))$ a sequence in $\Theta(K)$ converging to (μ, ν). Let $w_n = w_{\mu_n}$, $w = w_\mu$, and $h = w_\nu$. By Theorem 2B, $w_n \to w$ (in $C^\infty(\mathscr{U} \cup I)$). Morever, since $0\cdot w_n \circ f_n = \mu_n \cdot f_n \to \nu$ and since $w_n \circ f_n$ leaves $0, 1, \infty$ fixed, $w_n \circ f_n \to h$. It follows that $f_n \to f = w_n^{-1} \circ h$. Clearly $(\mu_n, f_n) \to (\mu, f) \in K$, and $(\mu, \nu) = \Theta(\mu, f) \in \Theta(K)$, completing the proof.

Remark. With more effort, one can prove that $\mathscr{D}(X)$ acts properly on $\mathscr{M}(X)$.

4. The fibre bundle, $e(X) < 0$

(A) To complete the proof of Theorem 1B we need to show that the quotient map $\Phi: \mathscr{M}(X) \to \mathscr{M}(X)/\mathscr{D}_0(X)$ has local cross-sections. For that purpose we first map $\mathscr{M}(X)$ into G^n, where G is the conformal automorphism group of $\mathscr{U}$, and $n = 1 - e(X)$ is the rank of the free group $\pi_1(X)$. Our assumption that $e(X) < 0$ remains in force.

Call (A, B) a *normalized pair* of Möbius transformations if each has two fixed points, the fixed points of A are at 0 and ∞, and the attractive fixed point of B is at 1.

Proposition. *Let x_0 be an interior point of X, and $c_1, \cdots, c_n$ a free system of generators for $\pi_1(X, x_0)$. For each conformal structure on X there exist a unique point $z_0 \in \mathscr{U}$ and holomorphic covering map $\pi: \mathscr{U} \to X_0$ so that*

(a) $\pi(z_0) = x_0$,

(b) *the cover transformations γ_1 and γ_2 determined by c_1, c_2, and z_0 are a normalized pair.*

The proof is the same as that of Lemma 4C in [9].

(B) For any $\mu \in \mathscr{M}(X)$, let $\pi: \mathscr{U} \to X_0$ be the covering map determined by Proposition 4A, and $\gamma_1, \cdots, \gamma_n$ the generators of the cover group Γ determined by the point $z_0 \in \mathscr{U}$ and the generators $c_1, \cdots, c_n$ of $\pi_1(X, x_0)$. We define $P: \mathscr{M}(X) \to G^n$ by $P(\mu) = (\gamma_1, \cdots, \gamma_n)$.

Let S be the set of points $(g_1, \cdots, g_n) \in G^n$ such that (g_1, g_2) is a normalized pair of Möbius transformations. Then P maps $\mathscr{M}(X)$ into S.

Lemma. *S is a locally closed real analytic submanifold of G^n of dimension $3n - 3 = -3e(X)$.*

We omit the easy proof.

(C) Now fix any point $\mu_0 \in \mathscr{M}(X)$ and let $\pi: \mathscr{U} \to X_0$ be determined by μ_0. The cover group Γ is generated by $s_0 = P(\mu_0) \in S$. Composing P with the inverse of $\pi^*: \mathscr{M}(X) \to \mathscr{M}(\Gamma)$, we obtain a map, still called $P: \mathscr{M}(\Gamma) \to S$.

Lemma. *$P(\mu) = w_\mu \circ s_0 \circ w_\mu^{-1}$ for all $\mu \in \mathscr{M}(\Gamma)$.*

Corollary. *$P(\mu_0) = P(\mu_1)$ if and only if μ_0 and μ_1 are $\mathscr{D}_0(X)$-equivalent. Thus, P induces an injection from $\mathscr{M}(X)/\mathscr{D}_0(X)$ into S.*

These are proved in the same way as the corresponding assertions in § 6 of [9].

(D) Let $Q(\Gamma)$ be the real vector space of functions φ holomorphic in $\mathscr{U} \cup I$, *real on I*, satisfying

$$(\varphi \circ \gamma)(\gamma')^2 = \varphi \qquad \text{for all } \gamma \in \Gamma .$$

$Q(\Gamma)$ is the lift to $\mathscr{U}$ of the space holomorphic quadratic differentials on X (with its given conformal structure μ_0) which are *real on ∂X*. The Riemann-Roch theorem tells us that the (real) dimension of $Q(\Gamma)$ is $-3e(X)$, the dimension of S. The next proposition is essentially due to Teichmüller (see [1], [5]).

Proposition. *$P: \mathscr{M}(\Gamma) \to S$ is continuous. The restriction of P to any finite dimensional affine subspace is real analytic. Moreover, the kernel of the differential $dP(0)$ at 0 is*

$$Q(\Gamma)^\perp = \left\{ \nu \in A^1(\Gamma);\ \mathrm{Im} \int_X \nu \varphi d\bar{z} \wedge dz = 0, \forall \varphi \in Q(\Gamma) \right\} .$$

Proof (see [8, Theorem 5]). The continuity and smoothness of P are consequences of Lemma 4C and [4, Theorem 11]. In addition, if $\gamma_\mu = w_\mu \gamma w_\mu^{-1}$ for $\gamma \in \Gamma$ and $\mu \in \mathscr{M}(\Gamma)$, then

$$\dot{\gamma}(\nu)(z) = \lim_{t\to 0} [\gamma_{t\nu}(z) - z]/t$$

exists for all $z \in \mathscr{U} \cup I$ and $\nu \in A^1(\Gamma)$. Further,

$$\dot{\gamma}(\nu) = f\circ\gamma - \gamma' f\ , \tag{4.1}$$

where f is real on I and satisfies $f_{\bar{z}} = \nu$ (see [3, p. 138] and [1]). If $\nu \in \operatorname{Ker} dP(0)$, then (4.1) vanishes for all $\gamma \in \Gamma$. Thus, if $\varphi \in Q(\Gamma)$, then $w = f\varphi dz$ is a 1-form on X and real on ∂X, and

$$\operatorname{Im} \int_X \nu\varphi d\bar{z} \wedge dz = \operatorname{Im} \int_X dw = 0\ ,$$

which proves $\operatorname{Ker} dP(0) \subset Q(\Gamma)^\perp$. But

$$-3e(X) = \dim S \geq \operatorname{codim} \operatorname{Ker} dP(0) \geq \operatorname{codim} Q(\Gamma)^\perp = \dim Q(\Gamma) = -3e(X)\ ,$$

so $Q(\Gamma)^\perp = \operatorname{Ker} dP(0)$.

Corollary. *P is an open continuous map with local sections.*

In fact, $dP(0)$ is surjective, where $P: \mathscr{M}(\Gamma) \to S$. But $0 \in \mathscr{M}(\Gamma)$ corresponds to $\mu_0 \in \mathscr{M}(X)$, which was chosen arbitrarily. The corollary is therefore an immediate consequence of the implicit function theorem.

(E) The *reduced Teichmüller space* $\mathscr{T}^\#(X)$ is the quotient space $\mathscr{M}(X)/\mathscr{D}_0(X)$. From Corollaries 4C and 4D we have the

Lemma. *$P: \mathscr{M}(X) \to S$ has the form $P = h\circ\Phi$, where $\Phi: \mathscr{M}(X) \to \mathscr{T}^\#(X)$ is the quotient map and $h: \mathscr{T}^\#(X) \to P(\mathscr{M}(X))$ is a homeomorphism.*

Thus, by Corollary 4D, $\Phi: \mathscr{M}(X) \to \mathscr{T}^\#(X)$ has local sections. Combining that fact with §§ 3D and 3E, we conclude that Φ defines a principal fibre bundle with structure group $\mathscr{D}_0(X)$. The proof of Theorem 1B is now complete.

We remark that the homeomorphism h from $\mathscr{T}^\#(X)$ onto the image of P induces a real analytic structure on $\mathscr{T}^\#(X)$.

(F) According to Teichmüller's Theorem [6], $\mathscr{T}^\#(X)$ is homeomorphic to a Euclidean space.

As in § 8C of [9], we obtain at once

Corollary 1. *The bundle $\Phi: \mathscr{M}(X) \to \mathscr{T}^\#(X)$ is topologically trivial.*

Corollary 2. *$\mathscr{M}(X)$ is homeomorphic to $\mathscr{T}^\#(X) \times \mathscr{D}_0(X)$. Thus $\mathscr{D}_0(X)$ is contractible.*

Corollary 2 gives us Theorem 1C for orientable surfaces X with $e(X) < 0$. The non-orientable surfaces will be considered in § 5.

5. Surfaces with symmetries

(A) We still assume that X is oriented and that $e(X) < 0$. It follows that for each $\mu \in \mathscr{M}(X)$ the subgroup of $\mathscr{D}(X)$ which leaves μ fixed is finite [16]. The converse is also true.

Lemma. *Let $H \subset \mathscr{D}(X)$ be a finite subgroup. Then*

$$\mathscr{M}(X)^H = \{\mu \in \mathscr{M}(X)\,;\ \mu \cdot h = \mu \text{ for all } h \in H\}$$

is a non-empty contractible submanifold of $\mathscr{M}(X)$.

Proof. Choose a Reimannian metric ρ on X, and view ρ as a quadratic form on the tangent space at each point. Then $\rho_0 = \sum (\rho \cdot h), h \in H$, is an H-invariant metric, and induces an H-invariant conformal structure on X. Thus $\mathscr{M}(X)^H$ is non-empty.

Now choose $\mu_0 \in \mathscr{M}(X)^H$ and let $\pi\colon \mathscr{U} \to X_0$ be a holomorphic covering map with cover group Γ. As in § 3, there exist an induced homeomorphism $\pi^*\colon \mathscr{M}(X) \to \mathscr{M}(\Gamma)$ and a group homomorphism $\pi_*\colon \mathscr{D}(\Gamma) \to \mathscr{D}(X)$. Let H' be the inverse image of H in $\mathscr{D}(\Gamma)$. Then π^* maps $\mathscr{M}(X)^H$ onto the H'-invariant elements of $\mathscr{M}(\Gamma)$. By construction, the usual conformal structure of $\mathscr{U}$ is H'-invariant, so H' is a subgroup of the automorphism group G of $\mathscr{U}$. Let H'_0 be the orientataion-preserving subgroup of H'. Then, for any $\mu \in \mathscr{M}(\Gamma)$,

$$\mu \cdot h = (\mu \circ h)\bar{h}'/h' \quad \text{if } h \in H'_0\,,$$
$$\overline{\mu \cdot h} = (\mu \circ h)\bar{h}_{\bar{z}}/h_{\bar{z}} \quad \text{if } h \in H' - H'_0\,.$$

It is clear from these formulas that the H'-invariant μ in $\mathscr{M}(\Gamma)$ form a contractible submanifold of $\mathscr{M}(\Gamma)$.

Corollary. *$\mathscr{D}_0(X)$ has no non-trivial subgroups of finite order.*

In fact, $\mathscr{D}_0(X)$ acts freely on $\mathscr{M}(X)$, so if H is a non-trivial subgroup of $\mathscr{D}_0(X)$, then $\mathscr{M}(X)^H$ is empty.

(B) Of course H acts on $\mathscr{D}_0(X)$ by the action

$$h \cdot g = hgh^{-1}, h \in H \quad \text{and} \quad g \in \mathscr{D}_0(X)\,.$$

The fixed point set $\mathscr{D}_0(X)^H$ is the subgroup of $\mathscr{D}_0(X)$ which maps $\mathscr{M}(X)^H$ into itself.

Let $\Phi\colon \mathscr{M}(X) \to \mathscr{T}^{\#}(X)$ and $\theta\colon \mathscr{D}(X) \to \mathscr{D}(X)/\mathscr{D}_0(X) = \Gamma(X)$ be the quotient maps. $\theta(H)$ is a finite subgroup of $\Gamma(X)$, isomorphic to H because of Corollary 5A. Of course the group $\Gamma(X)$ acts on $\mathscr{T}^{\#}(X)$, and the fixed point set $\mathscr{T}^{\#}(X)^{\theta(H)}$ includes $\Phi(\mathscr{M}(X)^H)$.

Theorem. *$\Phi\colon \mathscr{M}(X)^H \to \mathscr{T}^{\#}(X)^{\theta(H)}$ is an open surjective map, and defines a trivial principal fibre bundle with structure group $\mathscr{D}_0(X)^H$.*

(C) The proof of Theorem 5B will be divided into several steps. First we define a non-negative integer $d(H)$ as follows: Choose $\mu \in \mathscr{M}(X)^H$ and let $Q(X)$ be the corresponding space of holomorphic quadratic differential real on ∂X. Since H consists of holomorphic and conjugate holomorphic maps, relative to μ, H operates on $Q(X)$ as a group of linear transformations [13]. $d(H)$ is the dimension of the (real) subspace $Q(X)^H$ fixed by H. There are several ways to verify that $d(H)$ depends only on H; for instance we may appeal to the following important

Lemma. *$\mathscr{T}^\sharp(X)^{\theta(H)}$ is a closed connected subset of $\mathscr{T}^\sharp(X)$, homeomorphic to $\boldsymbol{R}^{d(H)}$.*

The lemma is due to Saul Kravetz [12, Lemma 5.1]. Kravetz considers only closed surfaces, but his proof applies equally well to our situation.

(D) **Lemma.** *Φ: $\mathscr{M}(X)^H \to \mathscr{T}^\sharp(X)^{\theta(H)}$ is open and continuous with local cross-sections.*

A proof of this lemma, again for closed surfaces, is given by Rauch in [13]. This time we provide some details. Choose any $\mu_0 \in \mathscr{M}(X)^H$ and form the corresponding covering π: $\mathscr{U} \to X_0$ and cover group Γ. Let $\rho(z)|dz|^2 = ds^2$ be the hyperbolic metric on $\mathscr{U} \cup I \cup \mathscr{U}^*$, and let $\varphi \in Q(\Gamma)$. If φ is close to zero, then $\bar{\varphi}\rho^{-1} \in \mathscr{M}(\Gamma)$. It follows from §§4B, C, and D that $\varphi \mapsto \Phi(\bar{\varphi}\rho^{-1})$ defines a diffeomorphism from a neighborhood N of 0 in $Q(\Gamma)$ onto a neighborhood of $\Phi(\mu_0)$ in $\mathscr{T}^\sharp(X)$. The intersection $N \cap Q(\Gamma)^H$ is mapped into $\mathscr{T}^\sharp(X)^{\theta(H)}$. The lemma follows, because $Q(\Gamma)^H$ and $\mathscr{T}^\sharp(X)^{\theta(H)}$ both have dimension $d(H)$.

(E) The rest of the proof is easy. Let $H' \subset \mathscr{D}(X)$ be a finite group, and write $H' \sim H$ if $\theta(H') = \theta(H)$. Then

$$\mathscr{T}^\sharp(X)^{\theta(H)} = \cup\ \Phi(\mathscr{M}(X)^{H'}), \quad H' \sim H\ .$$

Moreover, $\Phi(\mathscr{M}(X)^{H'})$ and $\Phi(\mathscr{M}(X)^H)$ are disjoint unless $H' = gHg^{-1}$, $g \in \mathscr{D}_0(X)$, when they coincide. Now Lemma 4D implies that $\Phi(\mathscr{M}(X)^H)$ is open, *hence closed*, in $\mathscr{T}^\sharp(X)^{\theta(H)}$, so $\Phi(\mathscr{M}(X)^H) = \mathscr{T}^\sharp(X)^{\theta(H)}$, by Lemma 4C. Since Φ is open and continuous, $\mathscr{T}^\sharp(X)^{\theta(H)}$ can be identified with the quotient $\mathscr{M}(X)^H/\mathscr{D}_0(X)^H$. Since Φ has local cross-sections, Φ defines a $\mathscr{D}_0(X)^H$-fibre bundle. By Lemma 4C, the base space of that bundle is contractible, so the bundle is trivial, and Theorem 5B is proved.

(F) **Proposition.** *If H is a finite subgroup of $\mathscr{D}_0(X)$, the group $\mathscr{D}_0(X)^H$ is contractible.*

Proof. By Theorem 5B, $\mathscr{D}_0(X)^H \times \mathscr{T}^\sharp(X)^{\theta(H)}$ is homeomorphic to the contractible space $\mathscr{M}(X)^H$.

Corollary. *Let Y be a non-orientable compact surface with boundary. If $e(Y) < 0$, then $\mathscr{D}_0(Y)$ is contractible.*

Proof. Let π: $X \to Y$ be a two-sheeted covering by the orientable surface X, and let $H \subset \mathscr{D}(X)$ be the cover group. $\mathscr{D}_0(Y)$ is homeomorphic to the contractible group $\mathscr{D}_0(X)^H$.

(G) **Remark.** The action $h \cdot g = hgh^{-1}$ of the finite group H on $\mathscr{D}_0(X)$ determines the pointed cohomology set $H^1(H, \mathscr{D}_0(X))$. The considerations of §5 E show that $H^1(H, \mathscr{D}_0(X))$ is trivial. In fact, $\theta(H) = \theta(H^1)$ if and only if $H^1 = gHg^{-1}$ for some $g \in \mathscr{D}_0(X)$.

6. The annulus and Möbius band

(A) Fix the point x_0 on the boundary of the annulus X, choose a simple loop c which generates $\pi_1(X, x_0)$, and put $I = \boldsymbol{R} - \{0\}$.

Lemma. *For each $\mu \in \mathscr{M}(X)$ there is a unique μ-conformal covering map $\pi: \mathscr{U} \cup I \to X$ so that*

(a) $\pi(x_0) = 1$,

(b) *the loop c determines a generator $\gamma(z) = kz$, $k > 1$, for the cover gronp Γ.*

As in § 3, π induces a map π^* from $\mathscr{M}(X)$ onto the space $\mathscr{M}(\Gamma)$ of Γ-invariant conformal structures on $\mathscr{U} \cup I$. Once again, we let $A^1(\Gamma)$ be the space of $\mu \in C^\infty(\mathscr{U} \cup I, C)$ such that

$$\mu \circ \gamma = \mu \,. \tag{6.1}$$

Proposition. *$\mathscr{M}(\Gamma)$ is the convex open set of all $\mu \in A^1(\Gamma)$ with* $\sup\{|\mu(z)|;$ $z \in \mathscr{U} \cup I\} < 1$, *and $\pi^*: \mathscr{M}(X) \to \mathscr{M}(\Gamma)$ is a diffeomorphism.*

Corollary. *$\mathscr{M}(X)$ is contractible.*

(B) Continuing by analogy with § 3, we let $\mathscr{D}_0(\Gamma)$ be the centralizer of Γ in $\mathscr{D}(\mathscr{U} \cup I)$ and $\mathscr{D}_0(\Gamma; 1)$ the subgroup fixing 1. Define $\pi_*: \mathscr{D}_0(\Gamma; 1) \to \mathscr{D}(X)$ by $\pi_*(f) \circ \pi = \pi \circ f$.

Proposition. *π_* is an isomorphism of $\mathscr{D}_0(\Gamma; 1)$ onto the group $\mathscr{D}_0(X; x_0)$ of diffeomorphisms of X, which fix x_0 and are homotopic to the identity.*

The proof is given in § 5B of [9], except that we use here the Γ-invariant metric $ds = |z|^{-1}|dz|$ on $C - \{0\}$.

(C) Once again we transfer the action of $\mathscr{D}_0(X; x_0)$ on $\mathscr{M}(X)$ by π to the action

$$\mu_f \cdot g = \mu_{f \circ g} \tag{6.2}$$

of $\mathscr{D}_0(\Gamma; 1)$ on $\mathscr{M}(\Gamma)$. Analogous to Propositions 5C and 5D of [9] we have

Proposition. *The action $\mathscr{M}(\Gamma) \times \mathscr{D}_0(\Gamma; 1) \to \mathscr{M}(\Gamma)$ by (6.2) is free, continuous, and proper.*

Corollary. *The natural action $\mathscr{M}(X) \times \mathscr{D}_0(X; x_0) \to \mathscr{M}(X)$ is free, continuous, and proper.*

(D) Define $P: \mathscr{M}(X) \to R^+$ by $P(\mu) = \log k$, where $\gamma(z) = kz$ is determined by Lemma 6A. We also denote by P the composed map $Po(\pi^*)^{-1}: \mathscr{M}(\Gamma) \to R^+$.

Lemma 1. *Let $P(0) = \log k_0$. Then*

$$P(\mu) = \log(w_\mu(k_0)) \qquad \textit{for all } \mu \in \mathscr{M}(\Gamma)\,.$$

Proof. $\pi_\mu = \pi \circ w_\mu^{-1}: \mathscr{U} \cup I \to X$ is the covering map determined by Lemma 6A, for all $\mu \in \mathscr{M}(\Gamma)$. Thus, $\gamma_0(z) = (\exp P(0))z$ and $\gamma_\mu(z) = (\exp P(\mu))z$ satisfy $\gamma_\mu = w_\mu \circ \gamma_0 \circ (w_\mu)^{-1}$.

Lemma 2. *$P(\mu) = P(\nu)$ if and only if μ and ν are $\mathscr{D}_0(\Gamma; 1)$-equivalent.*

Proof. We may assume $\nu = 0$, so $P(\mu) = P(\nu)$ if and only if w_μ commutes with γ_0; this happens if and only if $w_\mu \in \mathscr{D}_0(\Gamma; 1)$.

(E) **Proposition.** *$P: \mathscr{M}(\Gamma) \to R^+$ is continuous and surjective. Further, $\sigma: R^+ \to \mathscr{M}(\Gamma)$ defined by*

$$\sigma(t)(z) = \frac{t - \log k_0}{t + \log k_0} \frac{z}{\bar{z}} , \quad z \in \mathscr{U} \cup I ,$$

is a continuous cross-section of P.

Proof. To check that $P \circ \sigma$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ is the identity map we note that

$$w_{\sigma(t)}(z) = |z|^{\alpha - 1^z} ,$$

where $\alpha \log k_0 = t$.

Corollary. *P: $\mathscr{M}(\Gamma) \to \boldsymbol{R}^+$ is an open map.*

In fact a neighborhood of $0 \in \mathscr{M}(\Gamma)$ covers a neighborhood of $P(0)$ in $\boldsymbol{R}^+$. But $0 \in \mathscr{M}(\Gamma)$ corresponds to any $\mu_0 \in \mathscr{M}(X)$.

(F) Consolidating the above we obtain the following.

Theorem. *The quotient map*

$$\Phi \colon \mathscr{M}(X) \to \mathscr{T}^{\sharp}(X) = \mathscr{M}(X) / \mathscr{D}_0(X ; x_0)$$

defines a trivial principal fibre boundle, and $\mathscr{T}^{\sharp}(X)$ is homeomorphic to $\boldsymbol{R}^+$.

Corollary. *Let X be an annulus. Then $\mathscr{D}_0(X ; x_0)$ is contractible, and $\mathscr{D}_0(X)$ has the circle as strong deformation retract.*

(G) The theorem and corollary of § 6F are valid for the Möbius band as well as the annulus. For the proof we fix x_0 on the boundary of the Möbius band X and choose a simple loop c generating $\pi_1(X, x_0)$. All the results of §§ 6A, B, C, D, and E hold, provided we make these modifications:

1. In Lemma 6A, the cover group Γ is generated by $\gamma(z) = -k\bar{z}$, $k > 1$.
2. Formula (6.1) becomes $(\mu \circ \gamma) = \bar{\mu}$.
3. In § 6D, $P(\mu) = \log k$, where $\gamma(z) = -k\bar{z}$.
4. Lemma 1 of § 6D becomes $P(\mu) = -\log(-w_\mu(-k_0))$.

For emphasis, we repeat the proposition corresponding to Corollary 6F.

Proposition. *Let X be the Möbius band. Then $\mathscr{D}_0(X ; x_0)$ is contractible, and $\mathscr{D}_0(X)$ has the circle as strong deformation retract.*

The proof of Theorem 1C is now complete, modulo Theorem 2B.

7. Homotopy modulo the boundary

(A) Until further notice we assume that $e(X) < 0$, but we do not require X to be orientable. Let $\mathscr{D}_1(X)$ be the normal subgroup of $\mathscr{D}_0(X)$ consisting of the $f \in \mathscr{D}_0(X)$, which are homotopic to the identity modulo ∂X (holding ∂X pointwise fixed). Let π: $\mathscr{U} \cup I \to X$ be a covering map whose cover group Γ consists of conformal automorphisms of $\mathscr{U}$. As in § 3 there is an isomorphism π_* from the centralizer $\mathscr{D}_0(\Gamma)$ of Γ in $\mathscr{D}(\mathscr{U})$ onto $\mathfrak{D}_0(X)$. $\mathscr{D}_1(X)$ is the image under π_* of the group $\mathscr{D}_1(\Gamma)$ of maps $f \in \mathscr{D}_0(\Gamma)$, whose restriction to I is the identity. Let $\mathscr{D}(\Gamma, I)$ be the centralizer of Γ in the diffeomorphism group of I.

Proposition. *The restriction map*

$$res: \mathscr{D}_0(\Gamma) \to \mathscr{D}(\Gamma, I)$$

defines a trivial principal fibre bundle with fibre $\mathscr{D}_1(\Gamma)$.

Proof. Since $\mathscr{D}_1(\Gamma)$ is a closed subgroup of the topological group $\mathscr{D}_0(\Gamma)$, all we need is to define a continuous map

$$\sigma: \mathscr{D}(I, \Gamma) \to \mathscr{D}_0(\Gamma)$$

so that res $o\sigma$ is the identity. That is a simple matter; we shall outline the procedure.

Each interval I_j of I determines a noneuclidean halfplane H_j bounded by I_j and the noneuclidean line in $\mathscr{U}$ which joins the endpoints of I_j. Let H be the union of the H_j. For $f \in \mathscr{D}(I, \Gamma)$, we put $\sigma(f)$ equal to the identity in $\mathscr{U} - H$. Each H_j is mapped into itself by a cyclic subgroup Γ_j of Γ (H_j covers an annulus in X with one boundary curve on ∂X). The given map $f: I_j \to I_j$ commutes with Γ_j. We need to define σ so that $\sigma(f)$ maps H_j onto itself, equals the identity near the noneuclidean line which bounds H_j in $\mathscr{U}$, and commutes with Γ_j. We leave the construction to the reader.

Corollary. $\mathscr{D}_1(X)$ *is contractible.*

In fact $\mathscr{D}_0(X)$ is contractible because $e(X) < 0$.

(B) The proof of Theorem 1D when $e(X) \geq 0$ is a simple modification of the above argument. All that is necessary is to replace $\mathscr{D}_0(X)$ or its analog $\mathscr{D}_0(\Gamma)$ by a contractible subgroup. For the annulus or Möbius band the group $\mathscr{D}_0(X; x_0)$ suffices, as we saw in § 6. For the unit disk, we saw in § 2 that the group $\mathscr{D}_0(X; x_0, x_1, x_2)$ fixing three boundary points is appropriate. In any event, $\mathscr{D}_1(X)$ is a closed normal subgroup of the above groups and the homogeneous fibration is trivial, as in Proposition 7A. We conclude that $\mathscr{D}_1(X)$ is contractible in all cases, as Theorem 1D asserts.

8. The continuity theorem

(A) In this section we shall prove Theorem 2B. In fact, we shall prove the corresponding statement for functions of class $C^{m+\alpha}$, and first need some definitions.

Let D be a subregion of $\boldsymbol{R}^2$ bounded by smooth curves, and I an open subset of ∂D. For any integer $m \geq 0$ and real number $0 < \alpha < 1$, the Fréchet space $C^{m+\alpha}(D \cup I)$ is the vector space of complex valued functions on $D \cup I$, whose partial derivatives of order m satisfy uniform Hölder conditions with exponent α on each compact subset of $D \cup I$. Convergence in $C^{m+\alpha}(D \cup I)$ means convergence in the norm $\|\cdot\|^G_{m+\alpha}$ (see e.g. [15, pp. 6, 8]) on every compact set $G \subset D \cup I$.

If ∂D is compact, the Banach space $C^{m+\alpha}(\partial D)$ is the vector space of complex valued functions on ∂D, whose m^{th} order derivative (with respect to arc length) satisfies a uniform Hölder condition with exponent α on ∂D. We shall denote the usual norm by $\|\cdot\|^{\partial D}_{m+\alpha}$ (see e.g. [15, p. 18]).

Let us note two inequalities. If D is bounded and $f, g \in C^{m+\alpha}(\bar{D})$, then

$$\|fg\|^D_{m+\alpha} \leq C\|f\|^D_{m+\alpha}\|g\|^D_{m+\alpha}, \tag{8.1}$$

$$\|f\|^{\partial D}_{m+\alpha} \leq C\|f\|^D_{m+\alpha}, \tag{8.2}$$

where the number C depends on m, α, and D, but not on f or g.

(B) Let $D = \mathscr{U}$, and let $\mathscr{M}^{m+\alpha}(\mathscr{U} \cup I)$ be the set of functions $\mu \in C^{m+\alpha}(\mathscr{U} \cup I)$ such that $|\mu(z)| < 1$ for all $z \in \mathscr{U} \cup I$. If $|\mu(z)| \leq k < 1$ in $\mathscr{U}$, then there is a unique solution w_μ of Beltrami's equation (2.2) which is a homeomorphism of $\bar{\mathscr{U}}$ onto itself and leaves $0, 1, \infty$ fixed. If $\mu \in \mathscr{M}^{m+\alpha}(\mathscr{U} \cup I)$, then $w_\mu \in C^{m+1+\alpha}$ and is a C^{m+1} diffeomorphism onto its image. Theorem 2B is an immediate consequence of the following

Continuity theorem. *For each $k < 1$, the map $\mu \mapsto w_\mu$ is a homeomorphism of the set $\{\mu \in \mathscr{M}^{m+\alpha}(\mathscr{U} \cup I);\ \sup|\mu(z)| \leq k < 1, z \in \mathscr{U} \cup I\}$ onto its image in $C^{m+1+\alpha}(\mathscr{U} \cup I)$.*

Here the integer $m \geq 0$ and the number $0 < \alpha < 1$ are fixed but arbitrary. We remark that Ahlfors and Bers [4] have shown that the above map $\mu \mapsto w_\mu$ is continuous with respect to the compact-open topology in $C(\mathscr{U} \cup I)$. If there were no boundary segments our continuity theorem would be a consequence of the Ahlfors-Bers theorem and standard interior estimates (see [7]). The boundary estimates are harder to obtain. Our method yields an essentially self-contained proof of the complete continuity theorem. Of course we rely on the Ahlfors-Bers theorem.

(C) Since $(w_\mu)_z \neq 0$ in $\mathscr{U} \cup I$, the map $w_\mu \mapsto \mu$ is continuous. Thus, to prove the continuity theorem we need only show that $\mu \mapsto w_\mu$ is continuous. The proof will be given in three steps. We shall always assume that our functions $\mu(z)$ are bounded by a fixed number $k < 1$.

(C_1) *Step* 1. $D \subset\subset D_1$ will mean that $\bar{D}$ is a compact subset of $\boldsymbol{R}^2$ contained in D_1. By supp (f) we mean the closure of the set of points z where $f(z) \neq 0$. We shall first show that if $\mu_n \to 0$ in $\mathscr{M}^{m+\alpha}(\mathscr{U} \cup \boldsymbol{R})$, with

$$\operatorname{supp}(\mu_n) \subset G \subset\subset \mathscr{U} \cup \boldsymbol{R}$$

for some fixed G, then $w_{\mu_n} \to z$ in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$. It is obviously sufficient to show that $w_{\mu_n} \to z$ in $C^{m+1+\alpha}(\bar{G}_1)$ for any region G_1 for which $G \subset G_1 \subset\subset \mathscr{U} \cup \boldsymbol{R}$, where we may assume without loss of generality that G_1 is simply connected and ∂G_1 is of class C^∞.

We first remark that by a theorem of Ahlfors and Bers [4, p. 399], $w_{\mu_n} \to z$ uniformly on any compact subset of $\mathscr{U} \cup \boldsymbol{R}$. Extend each of the mappings w_{μ_n} to $\boldsymbol{R}^2$ as homeomorphisms by reflecting with respect to $\boldsymbol{R}$. Denoting the extended mappings by $w_{\hat{\mu}_n}$ we have

$$(8.3)\qquad w_{\hat{\mu}_n}(z) = \begin{cases} w_{\mu_n}(z) & \text{for } z \in \mathcal{U} \cup \boldsymbol{R}\,, \\ \overline{w}_{\mu_n}(\bar{z}) & \text{for } \bar{z} \in \mathcal{U}\,, \end{cases}$$

where it is easily verified that

$$(8.4)\qquad \hat{\mu}_n(z) = \begin{cases} \mu_n(z) & \text{for } z \in \mathcal{U} \cup \boldsymbol{R}\,, \\ \bar{\mu}_n(\bar{z}) & \text{for } \bar{z} \in \mathcal{U}\,. \end{cases}$$

We remark that $\hat{\mu}_n$ and the derivatives of $w_{\hat{\mu}_n}$ may have jump discontinuities across those points of $\boldsymbol{R}$ which belong to ∂G_1. Set $w_n = w_{\hat{\mu}_n}$; it follows from the above formulas that $w_n \to z$ uniformly on any compact subset of $\boldsymbol{R}^2$ and

$$\| \hat{\mu}_n \|_{m+\alpha}^{G_1} = \| \mu_n \|_{m+\alpha}^{G_1^*}\,, \qquad \| (w_n)_{\bar{z}} \|_{m+\alpha}^{G_1} = \| (w_n)_{\bar{z}} \|_{m+\alpha}^{G_1^*}\,,$$

where G_1^* is the reflected image of G_1. Let $G_2 = \{z;\, |z| < R\}$ where R is so large that $\overline{G}_1 \cup \overline{G}_1^* \subset G_2$, and set $A = G_2 - \overline{G}_1$. A is a doubly connected region with C^∞ boundary. Since $\hat{\mu}_n$ and $(w_n)_{\bar{z}}$ vanish outside $\overline{G}_1 \cup \overline{G}_1^*$ we have

$$(8.5)\qquad \|(w_n - z)_{\bar{z}}\|_{m+\alpha}^{A} = \|(w_n - z)_{\bar{z}}\|_{m+\alpha}^{G_1}\,.$$

We wish to estimate $\|w - z\|_{m+1+\alpha}^{G_1}$. For $z \in G_2$ the Pompeiu formula [15, p. 41] gives us the representation

$$(8.6)\qquad \begin{aligned} w_n(z) - z &= \frac{1}{2\pi i} \int_{|z|=R} \frac{(w_n(\xi) - \xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \iint_{G_1} \frac{(w_n(\xi) - \xi)_{\bar{\xi}}}{\xi - z} d\xi d\bar{\xi} \\ &\quad + \frac{1}{2\pi i} \iint_{A} \frac{(w_n(\xi) - \xi)_{\bar{\xi}}}{\xi - z} d\xi d\bar{\xi} \\ &= I_{1,n}(z) + I_{2,n}(z) + I_{3,n}(z)\,. \end{aligned}$$

From here on, C will denote a number which depends at most on m and α. Now

$$(8.7)\qquad \|I_{1,n}\|_{m+1+\alpha}^{G_1} \le C \,(\sup_{|z|=R} |w_n(z) - z|)\,.$$

The functions $I_{2,n}(z)$ and $I_{3,n}(z)$ are continuous on $\boldsymbol{R}^2$, and from classical estimates (see e.g. [15, p. 56])

$$(8.8)\qquad \|I_{2,n}\|_{m+1+\alpha}^{G_1} \le C\, \|(w_n - z)_{\bar{z}}\|_{m+\alpha}^{G_1}\,,$$

$$(8.9)\qquad \|I_{3,n}\|_{m+1+\alpha}^{A} \le C\, \|(w_n - z)_{\bar{z}}\|_{m+\alpha}^{A} = C\, \|(w_n - z)_{\bar{z}}\|_{m+\alpha}^{G_1}\,,$$

where we have used (8.5). By (8.2) and (8.9) we have

$$\|I_{3,n}\|_{m+1+\alpha}^{\partial G_1} \le C\, \|(w_n - z)_{\bar{z}}\|_{m+\alpha}^{G_1}\,.$$

But $I_{3,n}(z)$ is analytic in G_1 and continuous across ∂G_1; therefore (see e.g. [15, p. 22])

$$\|I_{3,n}\|^{G_1}_{m+1+\alpha} \leq C\,\|(w_n - z)_{\bar z}\|^{G_1}_{m+\alpha}\ . \tag{8.10}$$

Now $w_n(z) - z$ satisfies the non-homogeneous Beltrami equation

$$(w_n - z)_{\bar z} = \mu_n(w_n - z)_z + \mu_n \qquad \text{in } G_1\ .$$

Hence

$$\begin{aligned} \|(w_n - z)_{\bar z}\|^{G_1}_{m+\alpha} &\leq C(\|\mu_n\|^{G_1}_{m+\alpha}\,\|(w_n - z)_z\|^{G_1}_{m+\alpha} + \|\mu_n\|^{G_1}_{m+\alpha})\ , \\ \|(w_n - z)_{\bar z}\|^{G_1}_{m+\alpha} &\leq C(\|\mu_n\|^{G_1}_{m+\alpha}\,\|w_n - z\|^{G_1}_{m+1+\alpha} + \|\mu_n\|^{G_1}_{m+\alpha})\ . \end{aligned} \tag{8.11}$$

Applying the estimates (8.7), (8.8), (8.10) and (8.11) to (8.6) we obtain

$$\begin{aligned} \|w_n - z\|^{G_1}_{m+1+\alpha} \leq C\,(&\sup_{|z|=R}|w_n - z| + \|\mu_n\|^{G_1}_{m+\alpha}\,\|w_n - z\|^{G_1}_{m+1+\alpha} \\ &+ \|\mu\|^{G_1}_{m+\alpha})\ . \end{aligned} \tag{8.12}$$

Since $w_n \to z$ uniformly on any compact subset of $\boldsymbol{R}^2$ and $\|\mu_n\|^{G_1}_{m+\alpha} \to 0$, we have that $\|w_n - z\|^{G_1}_{m+1+\alpha} \to 0$ which was to be shown.

(C$_2$) *Step* 2. Suppose that $\mu_n \to \mu$ in $\mathscr{M}^{m+\alpha}(\mathscr{U} \cup \boldsymbol{R})$ where

$$\operatorname{supp}(\mu_n) \subset G \subset\subset \mathscr{U} \cup \boldsymbol{R}\ ,$$

for some fixed G. We wish to show that $w_{\mu_n} \to w_\mu$ in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$. The mappings $w_{\lambda_n} = w_{\mu_n} \circ w_\mu^{-1} \in C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$ are homeomorphisms of the closure of $\mathscr{U}$ onto itself fixing $0, 1, \infty$, where

$$\lambda_n = \left[\frac{\mu_n - \mu}{1 - \mu_n\bar\mu}\frac{(w_\mu)_z}{(\overline{w}_\mu)_{\bar z}}\right]\circ w_\mu^{-1}\ .$$

Since $(w_\mu)_z \in C^{m+\alpha}(\mathscr{U} \cup \boldsymbol{R})$, $(w_\mu)_z \neq 0$ on $\mathscr{U} \cup \boldsymbol{R}$ and $\sup\{|\mu_n\mu|;\ z \in \mathscr{U} \cup \boldsymbol{R}\} \leq k^2 < 1$, it follows easily that $\lambda_n \to 0$ in $\mathscr{M}^{m+\alpha}(\mathscr{U} \cup R)$. Since

$$\operatorname{supp}(\lambda_n) \subset w_\mu(G) \subset\subset \mathscr{U} \cup \boldsymbol{R}\ ,$$

we have from the result of Step 1 that $w_{\lambda_n} \to z$ in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$. Since precomposing w_{λ_n} with w_μ is a continuous operation in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$ we find that $w_{\mu_n} \to w_\mu$ in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$, which was to be shown.

(C$_3$) *Step* 3. We are now in a position to complete the proof of the continuity of the map $\mu \mapsto w_\mu$. Let $\mu \in \mathscr{M}^{m+\alpha}(\mathscr{U} \cup I)$ and suppose that $\mu_n \to \mu$ in $\mathscr{M}^{m+\alpha}(\mathscr{U} \cup I)$. We wish to show that $w_{\mu_n} \to w_\mu$ in $C^{m+1+\alpha}(\mathscr{U} \cup I)$. It is sufficient to show that for each point $z_0 \in \mathscr{U} \cup I$, there exists a neighborhood V of z_0 such that $w_{\mu_n} \to w_\mu$ in $C^{m+1+\alpha}(\overline{\mathscr{U} \cup I \cap V})$. Setting $w_{\mu_n} = w_n$ and $w_\mu = w$, we remark that $w_n \to w$ uniformly on any compact subset of $\mathscr{U} \cup \boldsymbol{R}$.

Let us first suppose that $z_0 \in I$. Let N_j be the open disk $N_j = \{z; |z-z_0|<jd\}$ where $j=1,2$ and $d>0$, and choose d so small that $\mu_n \to \mu$ in $C^{m+\alpha}(\overline{\mathscr{U} \cup I \cap N_2})$. Let $\beta(x)$ be a real valued C^∞ function of the real variable x defined for $x \geq 0$ with $0 \leq \beta(x) \leq 1$, $\beta(x) \equiv 1$ for $0 \leq x \leq d$ and $\beta(x) \equiv 0$ for $x \geq 2d$. Defining $\nu(z) = \beta(|z-z_0|)\mu(z)$ and $\nu_n(z) = \beta(|z-z_0|)\mu_n(z)$ for $z \in \mathscr{U} \cup \boldsymbol{R}$ we have that $\nu_n \to \nu$ in $\mathscr{M}^{m+\alpha}(\mathscr{U} \cup \boldsymbol{R})$ and $\operatorname{supp}(\nu_n) \subset \overline{\mathscr{U} \cup I \cap N_2}$. Setting $W_n = w_{\nu_n}$ and $W = w_\nu$ we have from Step 2 that $W_n \to W$ in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$; as a consequence $W_n^{-1} \to W^{-1}$ in $C^{m+1+\alpha}(\mathscr{U} \cup \boldsymbol{R})$.

Let $\hat{w}$, $\hat{w}_n$, $\hat{W}$, $\hat{W}_n$, $\hat{W}^{-1}$, $\hat{W}_n^{-1}$ be the homeomorphisms of $\boldsymbol{R}^2$ onto itself obtained by extending w, w_n, W, etc. by reflection with respect to $\boldsymbol{R}$. That is we define $\hat{w}$ etc. as in (8.3). It then follows that $\hat{w}_n \to w$ and $\hat{W}_n^{-1} \to \hat{W}^{-1}$ uniformly on $\overline{N}_2$ where $\hat{\mu}_n \equiv \nu_n$ and $\hat{\mu} \equiv \nu$ on $\overline{N}_1$. By the representation theorem of Morrey (see e.g. [15, p. 100])

$$\hat{w}_n(z) = \phi_n(\hat{W}_n(z)) \quad \text{on} \quad N_1\,, \tag{8.13}$$

where the ϕ_n are conformal mappings of the domains $\hat{W}_n(N_1)$ onto the domains $w_n(N_1)$. Now there exists a neighborhood N of z_0, $\overline{N} \subset N_1$, such that $\hat{W}(\overline{N}) \subset \hat{W}_n(N_1)$ for all n sufficiently large. Then since $\phi_n = \hat{w}_n \circ \hat{W}_n^{-1}$, it follows that the ϕ_n converge uniformly on $\hat{W}(\overline{N})$ and therefore the derivatives of ϕ_n of any finite order converge uniformly on any compact subset of $W(N)$. In view of (8.13), $w_n \to w$ in $C^{m+1+\alpha}(\overline{\mathscr{U} \cup I \cap V})$ where V is any neighborhood of z_0 with $\overline{V} \subset N$. Thus the proof is complete for $z_0 \in I$; for $z_0 \in \mathscr{U}$ we repeat the above argument, omitting the step in which the mappings are reflected.

Added and Proof

1) The paper of Z. G. Šeftel' [17] came to our attention recently. The continuity theorem of §8 can be derived from Theorem 1 of that paper together with interior estimates. We feel that our short self-contained proof has merit.

2) Since our continuity theorem deals with functions of class $C^{m+\alpha}$, we can construct an analogue of our bundle (1.1) with the Banach manifold of $C^{m+\alpha}$ conformal structures on X as total space. As a corollary, Theorem 1C remains true for diffeomorphisms of class $C^{m+1+\alpha}$, for $m \geq 0$.

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